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# Automorphisms of Automorphism Groups of Free Groups

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If $n \geq 3$ and $F_n$ is free of rank $n$, then $\mathrm{Out}(\mathrm{Aut}(F_n)) = \mathrm{Out}(\mathrm{Out}(F_n)) = \{1\}$.


## 1. INTRODUCTION

There is a well-developed analogy between lattices in semisimple Lie groups on the one hand and mapping class groups and automorphism groups of free groups on the other. A recently developed part of this analogy involves generalizations of the theorem of Tits [9] which states that all automorphisms of a spherical building stem from the underlying algebraic group. Specifically, Ivanov [5] proved that the natural map from the mapping class group to the group of automorphisms of the curve complex is an isomorphism, and the present authors proved that the natural map from $\mathrm{Out}(F_n)$ to the group of automorphisms of Outer space is an isomorphism. Tits's theorem leads to a proof that the outer automor-

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phism group of any irreducible lattice in higher rank is finite. In the same way, Ivanov's theorem implies that the outer automorphism group of the mapping class group is finite. In the case of $\mathrm{Out}(F_n)$ we have been unable to identify a direct implication of the same type. However, in this paper we shall show that some of the ideas that we used in [1] to analyze the symmetries of Outer space can also be used to prove:

THEOREM. *For $n \geq 3$, every automorphism of $\mathrm{Out}(F_n)$ is inner.*

The same techniques show that all automorphisms of $\mathrm{Aut}(F_n)$ are inner. In [4] Dyer and Formanek give a very different proof that $\mathrm{Out}(\mathrm{Aut}(F_n))$ is trivial. Their proof relies on the following classical result of Burnside: if the center of $G$ is trivial, then $\mathrm{Out}(\mathrm{Aut}(G)) = \{1\}$ if and only if $\mathrm{Inn}(G) \subset \mathrm{Aut}(G)$ is a characteristic subgroup.

*Outline of the Proof*

We shall assume throughout this paper that $n \geq 3$. We shall prove the theorem by considering the action of $\mathrm{Out}(F_n)$ on Outer space. (The corresponding result for $\mathrm{Aut}(F_n)$ is proved by an entirely similar analysis of the action of $\mathrm{Aut}(F_n)$ on Auter space.) The relevant facts and vocabulary concerning the geometry of Outer space can be found in [3].

We first consider roses in Outer space. The stabilizers of roses are characterized among the finite subgroups of $\mathrm{Out}(F_n)$ by their abstract isomorphism type, and hence there is a natural action of $\mathrm{Aut}(\mathrm{Out}(F_n))$ on the set of such stabilizers. Since $\mathrm{Out}(F_n)$ acts transitively on the set of roses, $\mathrm{Inn}(\mathrm{Out}(F_n))$ acts transitively on the set of stabilizers of roses. Thus, given an automorphism $\phi$ of $\mathrm{Out}(F_n)$, we may adjust $\phi$ within its outer automorphism class to ensure that $\phi$ leaves the stabilizer $W_n$ of the standard rose invariant. After further adjustment, we may assume that $\phi$ induces one of the small number of possible outer automorphisms of $W_n$ (Lemma 2.1). An analysis of the action of $\phi$ on the stabilizers of the graphs adjacent to the standard rose enables us to conclude that in fact the restriction of $\phi$ to $W_n$ is the identity (Lemma 3.3). We now observe that $\mathrm{Out}(F_n)$ is generated by $W_n$ together with one other automorphism $r$, which arises as a symmetry of a certain graph in the link of the standard rose. By analyzing the stabilizer of this graph we shall see that there are only a small number of possibilities for $\phi(r)$. A further analysis shows that each of the putative images $\phi(r) \neq r$ leads to inconsistent relations in $\mathrm{Out}(F_n)$. Thus we conclude that $\phi(r) = r$ and hence $\phi = 1$ in $\mathrm{Out}(\mathrm{Out}(F_n))$.

## 2. AUTOMORPHISMS OF THE STABILIZER OF A ROSE

Fix an ordered basis $\{x_1, \ldots, x_n\}$ of $F_n$. For $i = 1, \ldots, n$, let $e_i$ be the automorphism of $F_n$ that sends $x_i$ to $x_i^{-1}$ and fixes the other basis elements. For $i = 1, \ldots, n-1$, let $\tau_i$ be the automorphism that interchanges $x_i$ and $x_{i+1}$ while leaving the other basis elements fixed. Let $W_n \cong (\mathbb{Z}/2)^n \rtimes S_n$ be the subgroup of $\mathrm{Aut}(F_n)$ generated by the $e_i$ and the $\tau_i$. The group $\mathrm{Aut}(F_n)$ (and therefore $\mathrm{Out}(F_n)$) is generated by $W_n$ together with the automorphism $u$ which sends $x_1 \mapsto x_1 x_2$ and fixes $x_i$ for $i > 1$ (see [8], p. 163]).

There is a faithful representation $W_n \to GL(n, \mathbb{Z})$ whose image consists of all matrices of the form $PD$, where $P$ is a permutation matrix and $D$ is a diagonal matrix. The center of $W_n$ has order 2; it is generated by $z = \prod_{i=1}^n e_i$, which is represented by the matrix $-I$.

Define $\alpha: W_n \to W_n$ by $\alpha(PD) = (\det P)PD$, and $\beta: W_n \to W_n$ by $\beta(PD) = (\det D)PD$. It is easy to check that $\alpha$ is an automorphism for all $n$, and $\beta$ is an automorphism if and only if $n$ is even. Both $\alpha$ and $\beta$ have order 2, as does their product $\alpha\beta$. None of $\alpha$, $\beta$, or $\alpha\beta$ is inner, since an inner automorphism would preserve traces, while $\mathrm{tr}(\tau_1) \neq \mathrm{tr}(\alpha(\tau_1))$, $\mathrm{tr}(e_1) \neq \mathrm{tr}(\beta(e_1))$, and $\mathrm{tr}(\tau_1) \neq \mathrm{tr}(\alpha\beta(\tau_1))$.

We are grateful to Peter Neumann for his comments on the following lemma.

LEMMA 2.1. *If $n$ is odd, $\mathrm{Out}(W_n) \cong \mathbb{Z}/2$ is generated by $\alpha$. If $n$ is even, $\mathrm{Out}(W_n) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ is generated by $\alpha$ and $\beta$.*

*Proof.* Let $N = (\mathbb{Z}/2)^n$ be the normal subgroup generated by the $e_i$. We claim that $N$ is a characteristic subgroup of $W_n$. To see this, note that if $A = \phi(N) \neq N$ for some automorphism $\phi$ of $W_n$, then $A$ would have non-trivial image in the quotient $S_n = W_n/N$. Since the image would be a normal 2-group this is impossible unless $n = 4$ and the image is the Klein 4-group $V$. But the action of $V$ on $N$ has $z$ as its only fixed non-zero vector, contradicting the fact that the intersection of the abelian group $A$ with $N$ has order 4.

Since $N$ is characteristic, any automorphism of $W_n$ induces an automorphism of $S_n = W_n/N$. If $n \neq 6$, this automorphism must be inner, since $\mathrm{Out}(S_n)$ is trivial; we continue under the assumption that $n \neq 6$. Composing a given automorphism $\phi$ of $W_n$ with conjugation by an element of $S_n = 1 \rtimes S_n$, we may assume that for all $\sigma \in S_n$ we have $\phi(\sigma) = g_\sigma \sigma$ for some $g_\sigma \in N$.

Let $\mathrm{Fix}(\sigma)$ denote the fixed point set of the action of $\sigma \in S_n$ on $N$. For all $g \in N$ and $\sigma \in S_n$ we have $\phi(\sigma^{-1} g \sigma) = \sigma^{-1}\phi(g)\sigma$, so if $g \in \mathrm{Fix}(\sigma)$ then $\phi(g) \in \mathrm{Fix}(\sigma)$. Let $\sigma$ be the $n-1$ cycle $(2\,3 \ldots n)$. Since $\mathrm{Fix}(\sigma) = \{0, e_1, e_1 z, z\}$ and $\phi(e_1) \neq z$, we either have $\phi(e_1) = e_1$ or $\phi(e_1) = e_1 z$.

Given $i \neq 1$ we have $\phi(e_i) = \phi(e_1^{(1i)}) = \phi(e_1)^{(1i)}$ (writing the action of $S_n$ as exponentiation) and hence we either have $\phi(e_i) = e_i$ for $i = 1, \ldots, n$ or else $\phi(e_i) = e_i z$ for $i = 1, \ldots, n$. This latter possibility is fine if $n$ is even but does not define a homomorphism if $n$ is odd because in that case $\phi(z) = \sum_i z\phi(e_i) = z\sum_i \phi(e_i) = z^2 = 0$.

After possibly composing with the outer automorphism $\beta$ if $n$ is even, we may now assume that $\phi|_N$ is the identity, as well as that $\phi$ induces the identity on the quotient $W_n/N = S_n$; thus we are reduced to finding all splittings $\phi$: $S_n \to W_n$. Such splittings are in one-to-one correspondence with the elements of $H^1(S_n, N)$. Here $S_n$ acts on $N \cong (\mathbb{Z}/2)^n$ by permuting the factors, thus $N$ is induced from the trivial action of $S_{n-1}$ on $\mathbb{Z}/2$, that is $N = \mathrm{Ind}_{S_{n-1}}^{S_n}(\mathbb{Z}/2)$. By Shapiro's lemma we have $H^1(S_n, N) \cong H^1(S_{n-1}, \mathbb{Z}/2) \cong \mathbb{Z}/2$. Thus the only splittings $S_n \to W_n$ are the one fixing the generators $\tau_i$ and the one sending each $\tau_i$ to $z\tau_i$.

It only remains to see that the non-trivial outer automorphism of $S_6$ does not extend to an automorphism of $W_6$. But this is obvious from the fact that this outer automorphism interchanges the set of 3-cycles and the set of products of disjoint 3-cycles, and these different types of elements have stabilizers of different rank in the characteristic $N \subset W_6$. ∎

## 3. GEOMETRIC REALIZATIONS OF FINITE SUBGROUPS

In this section we shall make liberal use of the vocabulary associated with the action of $\mathrm{Out}(F_n)$ on Outer space. This action is related to the action of $\mathrm{Aut}(\mathrm{Out}(F_n))$ on the finite subgroups of $\mathrm{Out}(F_n)$ by the following facts.

Realization Theorem (See [2, 11]). Every finite subgroup of $\mathrm{Out}(F_n)$ fixes a vertex (marked graph) $[\Gamma, f]$ in the spine of Outer space, and there is a natural isomorphism from the stabilizer of $[\Gamma, f]$ to the group of graph-isomorphisms of $\Gamma$.

Unravelling the definitions, this amounts to the following: for every finite $G < \mathrm{Out}(F_n)$ one can find a connected graph $\Gamma$ (with no vertices of valence less than 3 and no separating edges), an isomorphism $f_*$: $F_n \to \pi_1(\Gamma)$ and a group $\tilde{G} = \{\tilde{g} \mid g \in G\}$ of automorphisms of $\Gamma$ such that $g = f_*^{-1} \circ \tilde{g}_* \circ f_*$ for all $g \in G$. We say that $[\Gamma, f]$ realizes $G$. (One can realize finite subgroups $G < \mathrm{Aut}(F_n)$ by graph automorphisms that fix a basepoint.)

*Stabilizers of Roses are Characteristic*

For $n \geq 3$, the subgroups of $\mathrm{Out}(F_n)$ isomorphic to $W_n$ are the maximal subgroups that can be realized by a rose; in other words, they are the stabilizers of the roses in Outer space (see, e.g., [7] or [10]).

LEMMA 3.1. *Let $\phi$ be an automorphism of* $\mathrm{Out}(F_n)$. *The outer automorphism class of $\phi$ contains an automorphism $\phi'$ such that $\phi'(W_n) = W_n$ and $\phi'|_{W_n} \in \{\mathrm{id}_{W_n}, \alpha, \beta, \alpha\beta\}$.*

*Proof.* We identify $W_n$ with the stabilizer in $\mathrm{Out}(F_n)$ of the rose $\rho_0$ associated with our fixed basis $x_1, \dots, x_n$ of $F_n$ ("the standard rose"). It follows from the second of the facts stated above that $\phi(W_n)$ is the stabilizer of some rose $\rho_1$ in Outer space. The action of $\mathrm{Out}(F_n)$ is transitive on roses, so $\rho_1 = \psi(\rho_0)$ for some $\psi \in \mathrm{Out}(F_n)$. Thus $\phi'(W_n) = W_n$, where $\phi'$ is the product of $\phi$ and $\mathrm{ad}(\psi) \in \mathrm{Inn}(\mathrm{Out}(F_n))$. Lemma 2.1 finishes the proof. ∎

We wish to eliminate the possibilities $\phi'|_{W_n} \in \{\alpha, \beta, \alpha\beta\}$. We shall do so by considering the action of $\phi'$ on the subgroup $H \subset W_n$ generated by $\{e_i, \tau_i \mid i = 3, \dots, n\}$ and $\tau_1$. The structure of $H$ is $H = W_{n-2} \times \mathbb{Z}/2$, where the factor $\mathbb{Z}/2$ is generated by $\tau_1$ and $W_{n-2}$ is generated by the remaining $\tau_i$ and $e_i$.

We shall also need the subgroup $G = W_{n-2} \times S_3$, where $W_{n-2}$ is as above and $S_3$ is the symmetric group of order 6 generated by $\tau_1$ and the automorphism

$$r = ue_2: \qquad x_1 \mapsto x_1 x_2^{-1} \quad \text{and} \quad x_2 \mapsto x_2^{-1}.$$

$G$ is the stabilizer of the marked graph $\gamma$ shown in Fig. 1, and $G \cap W_n = H$.

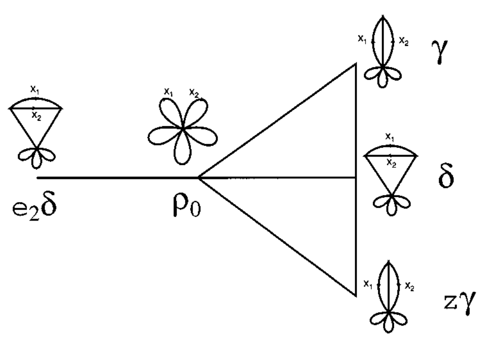


FIGURE 1.

LEMMA 3.2. *The fixed point set for the action of $H$ on the spine of Outer space consists of the subcomplex spanned by the vertices represented by the marked graphs shown in Fig.* 1, *and $\gamma$ is the unique point fixed by $G$.*

*Proof.* According to [6] we can find all graphs realizing $H$ by identifying one such graph and then successively collapsing and inserting $H$-invariant forests. (The process of inserting equivariant forests is called *equivalent blowing-up*.) One can perform an equivariant blowup of a graph $\Gamma$ that realizes $H$ if any only if the graph contains a set of oriented edges $I$ with the following properties:

(i) all edges in $I$ terminate at the same vertex $v$;

(ii) $I$ contains at least two and at most all but two of the edges terminating at $v$;

(iii) for $h \in H$, either $hI \cap I = I$ or $hI \cap I = \emptyset$;

(iv) if $v$ is a cut vertex, there are two edges $e$, $f$ terminating at $v$ which are in the same component of $\gamma - I$ such that $e \in I$ and $f \notin I$.

Since $\rho_0$ realizes $H$, we begin there. There are no invariant forests to collapse. There are four edge sets $I$ which meet the criteria (i)–(iv) above, namely $I_1 = \{x_1, x_2\}$, $I_2 = \{\bar{x}_1, \bar{x}_2\}$, $I_3 = \{\bar{x}_1, x_2\}$, and $I_4 = \{x_1, \bar{x}_2\}$. Blowing up $I_1$ results in the graph $\gamma$, blowing up $I_2$ results in $z\gamma$, and blowing up either $I_3$ or $I_4$ results in $e_2\delta$. The graph $\delta$ is obtained from $\gamma$ by blowing up $I_2$ (or from $z\gamma$ by blowing up $I_1$). No other graphs are obtainable by blowing up. Since collapsing invariant forests in these graphs produces no new marked graphs, this is the complete list of marked graphs fixed by $H$.

Any graph that realizes $G$ also realizes $H \subset G$. The automorphism groups of $\delta$ and $e_2\delta$ do not contain subgroups isomorphic to $G$, and $G$ is not contained in $W_n$, the stabilizer of $\rho_0$, nor in $z^{-1}Gz$, the stabilizer of $z\gamma$. Thus $G$ fixes only $\gamma$. ∎

LEMMA 3.3. *Every class* $[\phi] \in \mathrm{Out}(\mathrm{Out}(F_n))$ *has a representative* $\phi \in \mathrm{Aut}(\mathrm{Out}(F_n))$ *such that* $\phi|_{W_n} = \mathrm{id}_{W_n}$.

*Proof.* In the light of Lemma 3.1, we may assume that $\phi$ leaves $W_n$ invariant, and we will be done if we can show that the "bad" possibilities $\phi|_{W_n} \in \{\alpha, \beta, \alpha\beta\}$ do not arise. If $n \geq 4$, by arguing as in Lemma 3.2 we see that for each bad possibility the only graphs fixed by $\phi|_{W_n}(H)$ are $\rho_0$, $\delta$, and $e_2\delta$. Thus if there were an automorphism $\phi$ with bad $\phi|_{W_n}$ then one of $\{\rho_0, \delta, e_2\delta\}$ would have to realize $\phi(G) \supset \phi(H)$.

But $\phi(H) = \phi(G \cap W_n) = \phi(G) \cap W_n$ is a proper subgroup of $\phi(G)$, and $W_n$ is the stabilizer of $\rho_0$, so $\rho_0$ does not realize $\phi(G)$. And the automorphism groups of $\delta$ and $e_2\delta$ do not contain subgroups isomorphic to $G$.

It remains to consider the case $n = 3$, supposing that $\phi|_{W_n} = \alpha$ ($\beta$ does not give an automorphism of $W_n$ since $n$ is odd). In this case $\mathrm{Fix}(\phi(H))$ is the subcomplex spanned by $\rho_0$, $\delta$, $e_2\delta$, $e_2\gamma$, and $ze_2\gamma$, so it is possible that $\phi(G)$ is the stabilizer of either $e_2\gamma$ or $ze_2\gamma$. After conjugating by $z$ if necessary, we may assume $\phi(G) = e_2Ge_2$, the stabilizer of $e_2\gamma$.

The group $G$ is generated by $e_3$, $\tau_1$, and $r$. The only order 3 elements of $G$ are $r\tau_1$ and $\tau_1 r$, so that $\phi$: $G \to e_2Ge_2$ must take $r\tau_1$ to $e_2\tau_1 re_2$ or $e_2 r\tau_1 e_2$. Since we know that $\phi(\tau_1) = \alpha(\tau_1) = z\tau_1$ and $\phi(e_2) = e_2$, this implies that $\phi(r) = e_2e_3re_2$ or $e_2e_3\tau_1 r\tau_1 e_2$. In each of these cases we compute that, though $u = re_2$ (right multiplication of $x_1$ by $x_2$) commutes with $v = \tau_2 zuz\tau_2$ (left multiplication of $x_1$ by $x_3$), the values forced for $\phi(u)$ and $\phi(v)$ do not commute, even up to inner automorphism. Thus there is no outer automorphism $\phi$ which restricts to $\alpha$ on $W_3$. ∎

## 4. REMAINDER OF THE PROOF

In the previous section we showed that every class $[\phi] \in \mathrm{Out}(\mathrm{Out}(F_n))$ contains a representative $\phi$ such that $\phi|_{W_n}$ is the identity. We also saw that any graph realizing $\phi(G) \supset \phi(H) = H$ must be in $\mathrm{Fix}(H)$ and not equal to $\delta$, $e_2\delta$, or $\rho_0$. The only possibilities for such a graph are $\gamma$, in which case $\phi(G) = \mathrm{stab}(\gamma) = G$, and $z\gamma$, in which case $\phi(G) = \mathrm{stab}(z\gamma) = zGz^{-1}$. After composing by conjugation with $z$ if necessary, we may assume $\phi(G) = G$.

Recall that $G = W_{n-2} \times S_3$ that $H = W_{n-2} \times \langle \tau_1 \rangle$, and that $G$ is generated by $H$ and the element $r$ defined in Section 3. Recall also that $\mathrm{Out}(F_n)$ is generated by $W_n$ and $r$, so if we can show that $\phi(r) = r$ then we shall be done.

Since $\phi|_H$ is the identity, $\phi$ restricts to an automorphism of the centralizer of $W_{n-2}$ in $G$. This centralizer is $\mathbb{Z}/2 \times S_3$, where $\mathbb{Z}/2 = \langle z' \rangle$ is the center of $W_{n-2}$ (explicitly, $z' = e_3 \dots e_n$) and $S_3$ is generated by $r$ and $\tau_1$. The only order 3 elements of this centralizer are $r\tau_1$ and $\tau_1 r$, so that $\phi(r\tau_1) = r\tau_1$ or $\phi(r\tau_1) = \tau_1 r$. Since $\phi$ fixes $\tau_1$, this implies that $\phi(r) = r$ or $\phi(r) = \tau_1 r\tau_1$.

Suppose that $\phi(r) = \tau_1 r \tau_1$. We compute that $u = re_2$ commutes with $e_1ue_1$ but that $\phi(u) = \tau_1 r\tau_1 e_2$ does not commute with $\phi(e_1ue_1) = e_1\tau_1 r\tau_1 e_2 e_1$. Furthermore, the commutator of $\phi(u)$ and $\phi(e_1ue_1)$ is not inner, contradicting the fact that $\phi$ is a homomorphism. ∎

The proof that $\mathrm{Out}(\mathrm{Aut}(F_n))$ is trivial is essentially identical to the above proof; the only difference is that we consider each graph used as a graph with basepoint—the basepoint is at the (unique) vertex of maximal valence.

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